The Erdos-Rado Conjecture Implies Kalai's Second Question

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Abstract

We show that the Erdős-Rado conjecture implies a conjecture, which is implied in a question by Gil Kalai, for large n. More generally, we show $f(k,r,m,n) \leq 10f(k,r)^3\binom{n-m}{k-m}$ for $n \gg k$.

A r-*sunflower* is a family of sets $A_1, A_2, \ldots, A_r$ such that every element that belongs to more than one of the sets belongs to all of them. For a given family Y of sets and a set S, the *star* Z of Y is the subfamily of sets of Y, which contain S. We call the common elements to all elements of Y *head* of Y. If H is the head of Y, then *link* of S is the set $\{z \setminus H : z \in Z\}$.

A family of k-sets satisfies property $P(k,r,m)$ if it contains no sunflower with a head of size at most $m-1$. Let $f(k,r,m,n)$ the size of a largest family of k-sets in $\{1,2,\ldots,n\}$ with property $P(k,r,m)$. We denote $\lim_{n\to\infty} f(k,r,k,n)$ by $f(k,r)$. A family of k-sets satisfies property $Q(k,r,m)$ if it satisfies property $P(k,r,m)$ and if for all sets S of size at least m the link Y' of Y with respect to S has property $P(k-|S|,2,k-|S|)$ (i.e. we forbid intersection in at least m elements in Y). Let $g(k,r,m,n)$ be the size of a largest family of k-sets with property $Q(k,r,m)$. Notice that $g(k,r,k,n) = f(k,r,k,n) \leq f(k,r)$. If a family Y of k-sets in $\{1,2,\ldots,n\}$ satisfies property P and there exists no family Y' of k-sets in $\{1,2,\ldots,n\}$ satisfying property P with $Y \subsetneq Y'$, then we say that Y is a maximal family satisfying property P.

Conjecture 1 (Erdos-Rado Conjecture)**.** *For all r there exists a constant C_r such that*

$$f(k,r) \leq C_r^k.$$

In his first blog post[1] on the 10th POLYMATH project Gil Kalai asked several questions. Kalai's second question suggests the following conjecture.

Conjecture 2. *For all r there exists a constant C_r such that*

$$f(k,r,m,n) \leq C_r^k\binom{n-m}{k-m}.$$

This short note shows the following.

Theorem 3. *Let k,r,m,n be nonnegative integers such that $0 \leq m \leq k$ and $n \geq m(k-m)\binom{2k-m}{m+1}+m$. We have*

$$f(k,r,m,n) \leq 10g(k,r,m,n)^3\binom{n-m}{k-m} \leq 10f(k,r)^3\binom{n-m}{k-m}.$$

Proposition 4. *Let Y be a family of k-sets of $\{1,2,\ldots,n\}$ with property $P(k,r,m)$ and $n \geq m(k-m)\binom{2k-m}{m+1}+m$. Let Y_0 be a subset of Y, which is a maximal family satisfying property $Q(k,r,m)$. Define Y_1 by*

$$Y_1 = \{y \in Y : \text{ ex. } z_1, z_2 \in Y_0 \text{ with } |y \cap (z_1 \cup z_2)| > m\}.$$

Let $Y_2 = Y \setminus (Y_0 \cup Y_1)$. Then the following holds:

(a) $|Y_0| \leq g(k,r,m,n)$.

(b) $|Y_1| \leq g(k,r,m,n)^2\binom{n-m}{k-m}$,

[1] https://gilkalai.wordpress.com/2015/11/03/polymath10-the-erdos-rado-delta-system-conjecture/

(c) For all $y \in Y_2$ there exist a $z \in Y_0$ such that $|y \cap z| = m$.

Proof. Claim (a) is trivial.

We have $|Y_0|^2$ possibilities for choosing $z_1, z_2 \in Y_0$. At most $\binom{2k}{m+1}\binom{n-m-1}{k-m-1}$ sets y with k elements satisfy $|y \cap (z_1 \cup z_2)| > m$. For $n \geq (k-m)\binom{2k}{m+1} + m$ we have

$$\binom{2k}{m+1}\binom{n-m-1}{k-m-1} \leq \binom{n-m}{k-m},$$

so (b) follows.

Now let $y \in Y_2$.

If for all $z \in Y_0$ we have $|y \cap z| < m$, then $Y_0 \cup \{y\}$ has property $Q(k, r, m)$ (as Y has property $P(k, r, m)$). This contradicts the maximality of Y_0. Hence, we find a $z \in Y_0$ with $|y \cap z| \geq m$. If $|y \cap z| > m$, then $y \in Y_1$, so $|y \cap z| = m$. Hence, (c) follows. □

Proof of Theorem 3. Let Y be a family of k-sets of $\{1, 2, \ldots, n\}$ with property $P(k, r, m)$ as in Proposition 4. Define Y_0, Y_1, Y_2 as in Proposition 4.

For $z \in Y_0$ let Z^z be the set of elements of Y_2, which meet z in m elements. As Z^z satisfies property $P(k, r, m)$, we can choose $Z_0^z \subseteq Z^z$ as a maximal family satisfying property $Q(k, r, k)$. Define Z_1^z by

$$Z_1^z = \{y \in Z^z : \text{ ex. } z_1, z_2 \in Z_0^z \text{ with } |y \cap (z_1 \cup z_2)| > m\},$$

and Z_2^z as $Z \setminus (Z_0^z \cup Z_1^z)$. By Proposition 4 (a) and (b) we obtain

$$|Z_0^z| \leq g(k, r, m, n), \qquad\qquad |Z_1^z| \leq g(k, r, m, n)^2 \binom{n-m}{k-m}.$$

Let $z' \in Z_0^z$. Let $Z^{z,z'}$ be the set of elements of Z_2^z, which meet z' in exactly m elements. As $Z^{z,z'} \subseteq Z^z$, all $y \in Z^{z,z'}$ meet z and z' in m elements. As $z' \in Z^z$, $|z \cap z'| = m$. Hence, if $|y \cap z \cap z'| = i$, then $|y \cap (z \bigtriangleup z')| = 2(m-i)$. For given intersections of y with $z \cap z'$ and $z \bigtriangleup z'$, we have $\binom{n-2k+m}{k-2m+i}$ choices for y left. Hence,

$$\begin{aligned}
|Z^{z,z'}| &\leq \sum_{i=0}^{m} \binom{m}{i}\binom{2(k-m)}{2(m-i)}\binom{n-2k+m}{k-2m+i} \\
&= \binom{n-m}{k-m} + \sum_{i=0}^{m-1} \binom{m}{i}\binom{2(k-m)}{2(m-i)}\binom{n-2k+m}{k-2m+i} \\
&\leq \binom{n-m}{k-m} + m\binom{2k-m}{m+1}\binom{n-m-1}{k-m-1} \\
&\leq 2\binom{n-m}{k-m}
\end{aligned}$$

for $n \geq m(k-m)\binom{2k-m}{m+1} + m$.

We obtain

$$\begin{aligned}
|Z^z| &= |Z_0^z| + |Z_1^z| + |Z_2^z| \\
&\leq |Z_0^z| + |Z_0^z|^2\binom{n-m}{k-m} + |Z_0^z| \cdot 2\binom{n-m}{k-m} \\
&\leq g(k, r, m, n)\left(1 + (g(k, r, m, n) + 2)\binom{n-m}{k-m}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
|Y| &= |Y_0| + |Y_1| + |Y_2| \\
&\leq |Y_0| + |Y_0|^2\binom{n-m}{k-m} + |Y_0| \cdot g(k, r, m, n)\left(1 + (g(k, r, m, n) + 2)\binom{n-m}{k-m}\right) \\
&\leq 10g(k, r, m, n)^3\binom{n-m}{k-m} \leq 10f(k, r)^3\binom{n-m}{k-m}.
\end{aligned}$$

□

Corollary 5. *If the Erdős-Rado Conjecture is true, then Conjecture 2 is true for* $n \geq m(k-m)\binom{2k-m}{m+1} + m$.

Proof. If the Erdős-Rado conjecture is true, then for all r there exists a constant C_r such that $f(k,r) \leq C_r^k$. Let $C'_r = 10C_r^3$. By Proposition 4,

$$\begin{aligned} f(k,r,m,n) &\leq 10 f(k,r)^3 \binom{n-m}{k-m} \leq 10(C_r^3)^k \binom{n-m}{k-m} \\ &\leq 10(C'_r/10)^k \binom{n-m}{k-m} \leq C_r'^k \binom{n-m}{k-m}. \end{aligned}$$

□

Remark 6.

1. *It should be easy to improve many constants in the result.*
2. *I believe that very similar arguments should establish inequalities between* $f(k,r,m,n)$ *and* $f(k,r,m',n)$ *(instead of* $f(k,r,k,n)$*).*
3. *Considering* $g(k,r,m,n)$ *on its own might be interesting.*
4. *The argument is for large* n*. I would guess that simple arguments can extend the main result to something like* $n \geq \binom{2k-m}{m+1}$*. Anything beyond that would be very interesting.*