Partial m-Spreads of Hermitian Polar Spaces

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Abstract

A partial m-spread of maximals of a polar space is a set of maximal subspaces which covers each point at most m times. If each point is covered exactly m times, then a partial m-spread is an m-spread. We show that m-spreads of maximals do not exist in $H(2r-1, q^2)$ for r odd for $m < q$. This extends a result by Vanhove.

1 Introduction

The author visited John Bamberg and Jesse Landsdown at the University of Western Australia in November/December 2019. This document reports on one project which we persued, but which lacks substantial enough results to warrant publication.

A partial spread of the Hermitian polar space $H(2r-1, q^2)$ is a set of maximal isotropic subspaces such that no point of $H(2r-1, q^2)$ is covered more than once. Vanhove showed that a partial spread of the Hermitian polar space $H(2r-1, q^2)$, r odd, has size at most $q^r + 1$ [4]. Aguglia, Cossidente, and Ebert showed that there exist such partial spreads of size $q^r + 1$ [1]. A partial m-spread is a set of maximal isotropic subspaces such that no point of $H(2r-1, q^2)$ is covered more than m times. An m-spread is a partial m-spread such that each point is covered exactly m times. Write $[a]_q = (q^a - 1)/(q-1)$ for the number 1-spaces in $\mathbb{F}_q^a$. As $H(2r-1, q^2)$ possesses $(q^{2r-1} + 1)[r]_{q^2}$ points and each maximal isotropic subspace contains $[r]_{q^2}$ points, we find that an m-spread has size $m(q^{2r-1} + 1)$.

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Theorem 1. *Let $r \geq 3$ be an odd integer. A partial m-spread Y has size at most*

$$q^r + 1 + (m-1)[2r]_q.$$

Equality occurs only if $m < q$. In case of equality any two elements of Y are disjoint or meet in a point. Furthermore, each point lies on either (precisely) 0 or m elements of Y.

For $m \geq q$, the trivial bound of $m(q^{2r-1}+1)$ is better.

Corollary 2. *There exists no m-spread of $H(2r-1, q^2)$ for $m < q$.*

Equality occurs in Theorem 1 if and only if the points covered by the partial m-spread form a tight set. Hence, by Theorem 12 in [2], we have the following result.

Corollary 3. *If there exists a partial m-spread of size $q^r + 1 + (m-1)[2r]$, then there exists a strongly regular graph with parameters $(q^{4r}, i(q^{2r}-1), i(i-3) + q^{2r}, i(i-1))$, where $i = (q^r + 1 + (m-1)[2r]_q)/m = [2r]_q + \frac{q^r+1-[2r]_q}{m}$.*

Here i is an integer. Sometimes this improves the bound in Theorem 1 by 1, for instance for $(m, q) = (3, 4)$ we obtain

$$i = \frac{(2^{2r}+1)(2^{2r+1}+1)}{9}.$$

This number is an integer if only if $r \equiv 1 \pmod 3$. As we require r odd, here $r \equiv 1 \pmod 6$.

The strongly regular graphs obtained in this manner are fairly large. The smallest open case (in terms of partial m-spreads) is $(r, q, m) = (3, 3, 2)$ for which we obtain parameters $(531441, 142688, 38557, 38220)$.

2 Proof of the Bound

Let A_j denote the distance-j matrix of the dual polar graph associated with $H(2r-1, q^2)$ and let V_i denote the common eigenspaces auf the A_js. Then the eigenvalue of A_j belonging to V_i is, by [5, Theorem 4.3.6],

$$P_{ij} = \sum_{h=\max(i-j,0)}^{\min(r-j,i)} (-1)^{i-h}\, q^{(i-h)(i-h-1)+(j-i+h)^2} \begin{bmatrix} r-i \\ r-j-h \end{bmatrix}_{q^2} \begin{bmatrix} i \\ h \end{bmatrix}_{q^2}. \quad (1)$$

Here $\begin{bmatrix} a \\ b \end{bmatrix}_q$ denotes the number of b-spaces in $\mathbb{F}_q^a$.

Let $a = (a_0, \ldots, a_r)$ denote the inner distribution of a partial m-spread Y with characteristic vector χ, that is $a_i = |\{(A, B) \in Y : \dim(A \cap B) = r - i\}|/|Y|$. Let E_i be the orthogonal projection matrix onto V_i. The matrix E_i is positive semidefinite, so $\chi^T E_i \chi \geq 0$. Then $\chi^T E_i \chi \geq 0$ is equivalent to (for instance, see [5, Theorem 2.2.7])

$$\frac{P_{i0}}{P_{00}} a_0 + \frac{P_{i1}}{P_{01}} a_1 + \cdots + \frac{P_{ir}}{P_{0r}} a_r \geq 0.$$

We will apply this inequality for $i = r$. It follows from (1) that for $0 \leq j < r$, we have

$$\frac{P_{rj}}{P_{0j}} = -q \frac{P_{r,j+1}}{P_{0,j+1}}. \tag{2}$$

Furthermore, an m-spread satisfies by definition

$$[r-1]_{q^2} a_1 + [r-2]_{q^2} a_2 + \cdots + [2]_{q^2} a_{r-2} + a_{r-1} \leq (m-1)[r]_{q^2}. \tag{3}$$

Notice that $[n]_{q^2} - [n-1]_{q^2} = q^{2n-2} > q$ for $n \geq 2$. Hence, Equation (2), Equation (3) and r odd ensure that we maximize the sum of the a_j if $a_{r-1} = (m-1)[r]_{q^2}$ and $a_j = 0$ for $1 \leq j < r - 1$. Hence, we obtain that

$$1 + (m-1) q^{1-r} [r]_{q^2} - q^{-r} a_r \geq 0.$$

Hence, $a_r \leq q^r + (m-1) q [r]_{q^2}$. Hence,

$$|Y| \leq 1 + (m-1)[r]_{q^2} + q^r + (m-1) q [r]_{q^2} = q^r + 1 + (m-1)[2r]_q.$$

This also shows that equality can only occur when any two distinct elements of Y or disjoint or intersect in a point.

3 Constructions

We are unaware of any constructions for which Theorem 1 is tight for $1 < m < q$. The first open case is $(r, q, m) = (3, 3, 2)$.

In Theorem 4 of [3], Schmidt constructs a set of size q^{2r} of maximal isotropic subspaces of $H(2r-1, q^2)$ disjoint to one fixed maximal isotropic subspace. This set does not cover any point more than q times. Hence, we obtain a partial q-spread of $H(2r-1, q^2)$ of size $q^{2r} + 1$. Compare this to the size of q-spread which is $q(q^{2r-1} + 1) = q^{2r} + q$.

References

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