Explicit Ramsey Bounds

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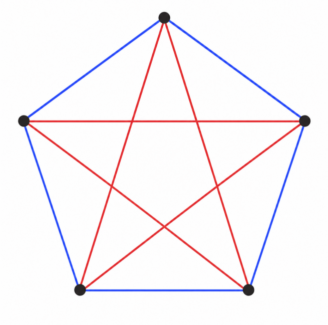
Ramsey Numbers (small)

Definition

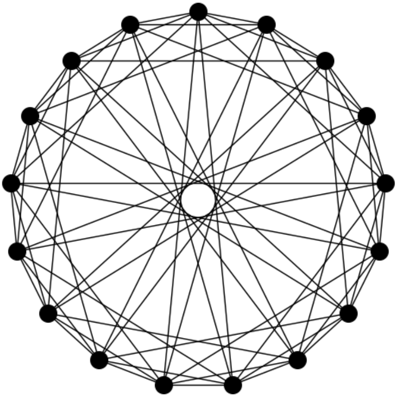
The **Ramsey number $R(s,t)$** denotes the smallest integer N such that any graph with at least N vertices either contains a **clique of size s** or an **independent set of size t**.



Witness for $R(3,3) \geq 6$. (tight!)

Paley Graphs

The Paley graph of order 16: $R(4,4) \geq 17$. (tight!)



For small s, t, very beautiful explicit lower bounds on $R(s,t)$!

Best First Order Bounds

Erdős-Szekeres (1935):

Theorem

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

For $s = t$, we have (**diagonal Ramsey numbers**)

$$\sqrt{2}^{(1+o(1))s} \leq R(s,s) \leq 3.8^{(1+o(1))s}$$

Erdős (1947); Campos et al. (2023), Gupta et al. (2024).

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For s fixed, we have (**off-diagonal Ramsey numbers**)

$$R(s,t) = t^{s-1-o(1)}.$$

Erdős-Szekeres (1935); Bradač (2026).

How do lower bounds go?

Erdős (1947) using **random deletion**:

$$R(s,s) \geq 2^{(\frac{1}{2}+o(1))s}.$$

Spencer (1975) slightly improved hidden constants using the **Lovász Local Lemma (LLL)**.

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Spencer, 1975: For off-diagonal, the LLL already gives

$$R(s,t) \geq t^{(s+1)/2-o(1)}.$$

What about **explicit constructions**?

Much, much worse!

Frankl-Wilson (diagonal)

Frankl-Wilson (1981):

$$R(s,s) \geq s^{(\frac{1}{4}-o(1))\log s/\log\log s}.$$

Alternative constructions due to Alon (1998) and Grolmusz (2000).

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Vertices: All (p^2-1)-subsets of $\{1,\ldots,p^3\}$.
Adjacency: $|A\cap B| \not\equiv -1 \pmod p$.

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Frankl-Wilson ***p*-rank bound**: cliques have size at most

$$\sum_{i=0}^{p-1} \binom{p^3}{i}.$$

Ray-Chaudhuri-Wilson **rank bound**: independent sets have size at most

$$\sum_{i=0}^{p-1} \binom{p^3}{i}.$$

Explcit versus strongly explicit

The Frankl-Wilson construction is **very concise** and **very explicit**.

Computer Science: **derandomization** just means **no randomness**.

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Barak, Rao, Shaltiel, Wigderson (2006/2012, Annals of Mathematics, 50 pages): a **polynomial time** construction showing

$$R(s,s) \geq 2^{(\log s)^{o(1)}}.$$

Xin Li (2023):

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Here we ignore this and focus on **strongly explicit**.

Alon-Pudlák (off-diagonal, general s)

The **norm graph** $NG_{q,r}$:

Vertices: vectors in $\mathbb{F}_{q^r}$.
Adjacency: a, b adjacent if

$$N(a+b) = 1,$$

where N is the norm from $\mathbb{F}_{q^r}$ to $\mathbb{F}_q$.

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Kollár, Rónyai, Szabó (1996): $K_{r,r!+1}$-free.

Alon-Pudlák (2001): k-clique graph of $N_{q,r}$ gives strongly explicit

$$R(s,t) \geq t^{\varepsilon \sqrt{\log s / \log \log s}}.$$

Kostochka, Pudlák, Rödl

Theorem (Alon (1994), KPR (2010))

Strongly explicit

$$R(3,t) = \Omega(t^{1.5}).$$

Alon's constructions uses **BCH codes**, KPR uses **projective planes**.

[1] One could extract such a bound from Alon-Pudlák, but this is tedious and needs s to be very, very large.

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Theorem (KPR (2010))

Strongly explicit

$$R(4,t) = \Omega(t^{1.6}), \qquad R(5,t) = \Omega(t^{1.\overline{6}}), \qquad R(6,t) = \Omega(t^{2}).$$

Question: For fixed s, when strongly explicit $R(s,t) \geq t^{2+\varepsilon}$?[1]

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Main Results

Theorem (Ih.-Mattheus (2026*))

Strongly explicit construction showing that

$$R(s,t) \geq t^{(1-o(1))\lceil \log s \rceil / \log(\lceil \log s \rceil + 1)}.$$

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Kostochka, Pudlák, Rödl (2010) $\longrightarrow$ Ih.-Mattheus:

$$R(\mathbf{6},t) \geq t^{\mathbf{2}-o(1)} \longrightarrow R(\mathbf{33},t) \geq t^{\mathbf{2.1}-o(1)}.$$

The Construction of $Tr^{\prec}_{d,h}$

Technicalities (ignore):

- Put $q = 2^h$. Let $d \geq 3$. Consider $\mathbb{F}_{q^d}$ as a d-dimensional vector space over $\mathbb{F}_q$.
- Let $\beta \in \mathbb{F}_{q^d}$ be normal over $\mathbb{F}_q$, that is,

$$\beta, \beta^q, \ldots, \beta^{q^{d-1}}$$

  is a basis of $\mathbb{F}_{q^d}$. Put $a = \beta + \beta^q$.
- Also, Tr is the trace from $\mathbb{F}_{q^d}$ to $\mathbb{F}_q$.
- Also, pick some ordering $\prec$ on the points of $\mathbb{F}_{q^d}$.

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We had a general idea for a construction and asked the ChatGPT 5.6 Plus chat for examples. It gave a graph with the same vertices, but **adjacency if $Tr(ax/y) = 0$ and $Tr(ay/x) = 0$** which is slightly worse.

The Independence Number

A variant of the following was the **main idea** (before asking ChatGPT).

Theorem

The independence number of $Tr^{\prec}_{d,h}$ is at most $d^h + 1$.

Proof.

The point-hyperplane incidence matrix B of $PG(d-1, 2^h)$ has **2-rank $d^h + 1$**.

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The point-hyperplane incidence matrix B of $PG(d-1, 2^h)$ has **2-rank $d^h + 1$**. An independent set corresponds to a full-rank principal submatrix of B. $\square$

Very similar to the p-rank bound in Frankl-Wilson (1981).

The Clique Bound

A variant of the argument below ChatGPT gave for free with the construction.

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- Parts of a clique live in a **Segre variety** $\mathcal{S}_{(r-1)\times(d-r-1)}$.

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- The Segre variety and the hyperplanes are in **general position** (here the choice of $a = \beta + \beta^q$ matters).
- Multilinear **Bezóut's theorem** gives the bound. $\square$

Again, this is a kind of **rank argument**.

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Open Problem

Problem (Erdős)

Explicitly show that

$$R(s,s) \geq C^s$$

for some $C > 1$.

Problem (Alon-Pudlák/Ih.-Mattheus)

Strongly explicitly show, for fixed s*, that*

$$R(s,t) \geq t^{\Omega(s^\varepsilon)}.$$

Alon-Pudlák only asked for explicit, but left out the ε.

Thank you for your attention!